

EIGENVALUES OF THE FRACTIONAL LAPLACE OPERATOR IN THE INTERVAL

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ABSTRACT. Two-term Weyl-type asymptotic law for the eigenvalues of one-dimensional fractional Laplace operator $(-\Delta)^{\alpha/2}$ ($\alpha \in (0, 2)$) in the interval $(-1, 1)$ is given: the n -th eigenvalue is equal to $(n\pi/2 - (2 - \alpha)\pi/8)^\alpha + O(1/n)$. Simplicity of eigenvalues is proved for $\alpha \in [1, 2)$. L^2 and L^∞ properties of eigenfunctions are studied. We also give precise numerical bounds for the first few eigenvalues.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Let $D = (-1, 1)$ and $\alpha \in (0, 2)$. Below we study the asymptotic behavior of the eigenvalues of the following spectral problem:

$$\left(-\frac{d^2}{dx^2}\right)^{\alpha/2} \varphi(x) = \lambda \varphi(x), \quad x \in D, \quad (1)$$

where $\varphi \in L^2(D)$ is extended to \mathbf{R} by 0 (for details, see below). It is known that there exist an infinite sequence of eigenvalues λ_n , $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$, and the corresponding eigenfunctions φ_n form a complete orthonormal set in $L^2(D)$. The following is the main result of this article.

Theorem 1. *We have*

$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(2 - \alpha)\pi}{8}\right)^\alpha + O\left(\frac{1}{n}\right). \quad (2)$$

More precisely, there are absolute constants C, C' such that

$$\left|\lambda_n - \left(\frac{n\pi}{2} - \frac{(2 - \alpha)\pi}{8}\right)^\alpha\right| \leq \frac{C(2 - \alpha)}{\sqrt{\alpha}} \frac{1}{n}$$

for $n \geq (C'/\alpha)^{3/(2\alpha)}$.

The scaling property of the fractional Laplace operator $(-d^2/dx^2)^{\frac{\alpha}{2}}$ implies that $\lambda_n(kD) = k^{-\alpha}\lambda_n(D)$. Hence, one easily finds the asymptotic formula for any interval.

By following carefully the proof, one can take e.g. $C = 30\,000$ and $C' = 4\,000$ above. Note that the constant in the error term $O(1/n)$ tends to zero as α approaches 2, and in the limiting case $\alpha = 2$ (not considered below), we have $\lambda_n = (n\pi/2)^2$ without an error term. A stronger version of Theorem 1 for $\alpha = 1$ was proved in [13].

The proof of Theorem 1 is modelled after [13]. In Section 2, an estimate for the fractional Laplace operator is given. The formula for the eigenfunctions on the half-line from [14] is recalled and studied in Section 3. An approximation to eigenfunctions is given in Section 4,

Work supported by the Polish Ministry of Science and Higher Education grant no. N N201 373136.

α	λ_1		λ_2		λ_3	
0.01	0.998	<i>0.997</i>	1.009	<i>1.009</i>	1.014	<i>1.014</i>
0.1	0.981	<i>0.973</i>	1.091	<i>1.092</i>	1.147	<i>1.148</i>
0.2	0.971	<i>0.957</i>	1.195	<i>1.197</i>	1.319	<i>1.320</i>
0.5	0.991	<i>0.970</i>	1.598	<i>1.601</i>	2.029	<i>2.031</i>
1	1.178	<i>1.158</i>	2.749	<i>2.754</i>	4.316	<i>4.320</i>
1.5	1.611	<i>1.597</i>	5.055	<i>5.059</i>	9.592	<i>9.597</i>
1.8	2.056	<i>2.048</i>	7.500	<i>7.501</i>	15.795	<i>15.801</i>
1.9	2.248	<i>2.243</i>	8.594	<i>8.593</i>	18.710	<i>18.718</i>
1.99	2.444	<i>2.442</i>	9.733	<i>9.729</i>	21.820	<i>21.829</i>

TABLE 1. Comparison of the approximation $\tilde{\lambda}_n = (\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8})^\alpha$ (roman font), and numerical approximations to λ_n obtained using the method of [15] with 5000×5000 matrices (slanted font).

Theorem 1 is proved in Section 5, and three further properties of eigenfunctions and eigenvalues are studied in Section 6. Sections 4–6 correspond to Sections 8–10 in [13]. Proposition 3 gives the simplicity of the eigenvalues when $\alpha \in (1, 2)$. The result follows relatively easily from the result for $\alpha = 1$ in [13]. In Propositions 1 and 2, $L^2(D)$ and $L^\infty(D)$ bounds for eigenfunctions are given. Finally, in Section 7, numerical estimates of λ_n in terms of eigenvalues of large dense matrices are obtained.

First-term Weyl-type asymptotic for λ_n was proved by Blumenthal and Getoor in 1959 [3]. The best known general estimate for λ_n is $\frac{1}{2}(\frac{n\pi}{2})^\alpha \leq \lambda_n \leq (\frac{n\pi}{2})^\alpha$ due to DeBlassie [9] and Chen and Song [7]. The important case of $\alpha = 1$ was studied in detail by several authors, see [1, 13] and the references therein. It is known that $(\lambda_n)^{1/\alpha}$ is continuous and increasing in $\alpha \in (0, 2]$, see [7, 8, 9, 10]. For a discussion of related results and historical remarks, see e.g. [1, 13]. Theorem 1 is of interest in physics, the asymptotic formula (2) (without the information about the order of the error term) was supported by numerical experiments in [15], and there is a considerable amount of related (mostly numerical) research in physics literature.

Noteworthy, although the values of C and C' given above are rather large, numerical evidence suggests that the error term in formula (2) is rather small also for small n in the full range of $\alpha \in (0, 2)$, see Table 1 and the estimates in the last section of this article. It is an interesting open problem to prove Theorem 1 with C and C' non-exploding as α approaches 0. This is related to simplicity of eigenvalues λ_n , conjectured to hold for all $\alpha \in (0, 2)$, proved for $\alpha = 1$ in [13], and extended to $\alpha \in [1, 2)$ in Proposition 3 in Section 6.

Motivated by the results of [13] and [14], as well as by Theorem 1 above, one can conjecture asymptotic law similar to (2) for eigenvalues on an interval for more general operators $\mathcal{A} = \psi(-d^2/dx^2)$, studied in [14]. While such a result for each individual ψ should present no difficulty (under some reasonable assumptions on the growth of ψ at infinity), it is an interesting (and much more difficult) problem to obtain estimates uniform also in ψ , for a given class of ψ . One important example here is the family of Klein-Gordon square-root operators $\mathcal{A} = \sqrt{m^2 - d^2/dx^2} - m$, with mass m ranging from 0 to ∞ . This operator is close to $\sqrt{-d^2/dx^2}$ for small m , but when m is large, it more similar to $-d^2/dx^2$.

To give a formal statement of the spectral problem (1), we recall the definition of the one-dimensional fractional Laplace operator $\mathcal{A} = (-d^2/dx^2)^{\alpha/2}$. It is defined pointwise by the principal value integral, if convergent,

$$\mathcal{A}f(x) = c_\alpha \text{pv} \int_{-\infty}^{\infty} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} dy, \quad x \in \mathbf{R}, \quad (3)$$

where

$$c_\alpha = \frac{2^\alpha \Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi} |\Gamma(-\frac{\alpha}{2})|};$$

$\mathcal{A}f(x)$ is convergent if, for example, f is smooth in a neighborhood of x and bounded on \mathbf{R} . Note that

$$\frac{1}{8}\alpha(2 - \alpha) \leq c_\alpha \leq \frac{1}{2}\alpha(2 - \alpha). \quad (4)$$

For $f \in C_c^\infty(\mathbf{R})$, the Fourier transform of $\mathcal{A}f$ is equal to $|\xi|^\alpha \hat{f}(\xi)$, and \mathcal{A} extends to an unbounded self-adjoint operator on $L^2(\mathbf{R})$. We write \mathcal{A}_D for the operator \mathcal{A} on D with zero exterior condition on $\mathbf{R} \setminus D$. More precisely, for $f \in C_c^\infty(D)$, $\mathcal{A}_D f$ is defined to be the restriction of $\mathcal{A}f$ to D . Again, \mathcal{A}_D extends to an unbounded self-adjoint operator on $L^2(D)$.

The operator $-\mathcal{A}$ (on an appropriate domain) is the generator of the one-dimensional symmetric α -stable process X_t , and $-\mathcal{A}_D$ is the generator of X_t killed upon leaving the interval D . This probabilistic interpretation is a primary source of our motivation, but will not be exploited in the sequel.

Notation. Throughout this article, C denotes an absolute constant (independent of α). We will track the dependence of other constants employed below on α to catch their asymptotic behavior as $\alpha \searrow 0$ and $\alpha \nearrow 2$. For brevity, we denote $\beta = 2 - \alpha$.

2. AUXILIARY ESTIMATES

Define, as in [13], Appendix C, an auxiliary function:

$$q(x) = \begin{cases} 0 & \text{for } x \in (-\infty, -\frac{1}{3}), \\ \frac{9}{2}(x + \frac{1}{3})^2 & \text{for } x \in (-\frac{1}{3}, 0), \\ 1 - \frac{9}{2}(x - \frac{1}{3})^2 & \text{for } x \in (0, \frac{1}{3}), \\ 1 & \text{for } x \in (\frac{1}{3}, \infty). \end{cases} \quad (5)$$

Note that q is piecewise C^2 , and $q(x) + q(-x) = 1$. Fix a piecewise C^2 function f on \mathbf{R} , and let $g(x) = q(x)f(x)$. Further, we assume that the support of g is compact. Below we estimate $\mathcal{A}g$ on $(-1, 0)$ in a very similar way as in [13].

Choose M to be the supremum of $\max(|f(x)|, |f'(x)|, |f''(x)|)$ over $x \in (-\frac{1}{3}, \frac{1}{3})$. Let $I = \int_0^\infty |f(x)|dx$. Then

$$|g''(x)| \leq |f(x)q''(x)| + 2|f'(x)q'(x)| + |f''(x)q(x)| \leq CM.$$

Suppose first that $x \in (-1, -\frac{1}{3})$. Since g vanishes in $(-1, -\frac{1}{3})$, $c_\alpha^{-1}|\mathcal{A}g(x)|$ is bounded above by

$$\begin{aligned} \int_{-\frac{1}{3}}^{\infty} \frac{|g(y)|}{|x-y|^{1+\alpha}} dy &\leq M \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{q(y)}{|x-y|^{1+\alpha}} dy + \frac{3^{1+\alpha}}{2^{1+\alpha}} \int_{\frac{1}{3}}^{\infty} |f(y)| dy \\ &\leq \frac{2^{1-\alpha} 3^\alpha M_0}{2-\alpha} + \frac{3^{1+\alpha} I}{2^{1+\alpha}} \leq \frac{CM}{\beta} + CI. \end{aligned}$$

In the second inequality we used the estimate $q(x)/|x-z|^{1+\alpha} \leq \frac{9}{2}(x+\frac{1}{3})^{1-\alpha}$. For $x \in (-\frac{1}{3}, 0)$ the principal value integral in the definition of \mathcal{A} can be estimated by splitting it into two parts. By Taylor's expansion of g , we have

$$\begin{aligned} \left| \text{pv} \int_{x-\frac{1}{3}}^{x+\frac{1}{3}} \frac{g(x) - g(y)}{|x-y|^{1+\alpha}} dy \right| &\leq \sup \left\{ \frac{1}{2} |g''(y)| : y \in (x-\frac{1}{3}, x+\frac{1}{3}) \right\} \int_{x-\frac{1}{3}}^{x+\frac{1}{3}} \frac{(x-y)^2}{|x-y|^{1+\alpha}} dy \\ &\leq \frac{\sup \{|g''(y)| : y \in (-\frac{1}{3}, \frac{1}{3})\}}{3^{2-\alpha}(2-\alpha)} \leq \frac{CM}{\beta}. \end{aligned}$$

Here for the second inequality note that $g''(y) = 0$ for $y < -\frac{1}{3}$. Furthermore,

$$\begin{aligned} &\left| \left(\int_{-\infty}^{x-\frac{1}{3}} + \int_{x+\frac{1}{3}}^{\infty} \right) \frac{g(x) - g(y)}{|x-y|^{1+\alpha}} dy \right| \\ &\leq |g(x)| \left(\int_{-\infty}^{x-\frac{1}{3}} + \int_{x+\frac{1}{3}}^{\infty} \right) \frac{1}{(x-y)^{1+\alpha}} dy + 3^{1+\alpha} \int_{x+\frac{1}{3}}^{\infty} |f(y)| dy \leq \frac{CM}{\alpha} + CI. \end{aligned}$$

We conclude that

$$c_\alpha^{-1} |\mathcal{A}g(x)| \leq \frac{CM}{\alpha\beta} + CI, \quad x \in (-1, 0). \quad (6)$$

3. ESTIMATES FOR HALF-LINE

The main result of [14] is the formula for generalized eigenfunctions for a class of operators on $(0, \infty)$. The case of fractional Laplace operator is studied in [14], Example 1. In particular, the eigenfunction F_λ of $\mathcal{A}_{(0,\infty)}$ corresponding to the eigenvalue λ^α ($\lambda > 0$) is shown to be $F_\lambda(x) = F(\lambda x) = \sin(\lambda x + \frac{\beta\pi}{8}) - G(\lambda x)$ (recall that $\beta = 2 - \alpha$), where G is a completely monotone function. More precisely, G is the Laplace transform of

$$\gamma(s) = \frac{\sqrt{2\alpha} \sin(\frac{\alpha\pi}{2})}{2\pi} \frac{s^\alpha}{1 + s^{2\alpha} - 2s^\alpha \cos(\frac{\alpha\pi}{2})} \exp \left(\frac{1}{\pi} \int_0^\infty \frac{1}{1+r^2} \log \frac{1-r^\alpha s^\alpha}{1-r^2 s^2} dr \right). \quad (7)$$

Furthermore, by [14], Lemma 13, we have

$$G(s) \leq \sin(\frac{\beta\pi}{8}) \leq C\beta, \quad (8)$$

and

$$\int_0^\infty G(s) ds = \cos(\frac{\beta\pi}{8}) - \sqrt{\frac{\alpha}{2}} \leq C\beta. \quad (9)$$

Note that the exponent in (7) is negative. Furthermore, for $\alpha \in (0, 1]$ we have

$$1 + s^{2\alpha} - 2s^\alpha \cos(\frac{\alpha\pi}{2}) \geq (\sin(\frac{\alpha\pi}{2}))^2 \geq \alpha^2,$$

while for $\alpha \in (1, 2)$, the left hand side is not less than one. Hence, for all $\alpha \in (0, 2]$,

$$1 + s^{2\alpha} - 2s^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \geq \min(\alpha^2, 1) \geq \frac{\alpha^2}{4}.$$

Finally, $\sin(\frac{\alpha\pi}{2}) \leq \alpha(2 - \alpha) = \alpha\beta$. Therefore,

$$\gamma(s) \leq \frac{2\sqrt{2\alpha}\beta}{\alpha\pi} s^\alpha. \quad (10)$$

By direct integration of the Laplace transform, we obtain that

$$G(s) \leq \frac{2\sqrt{2\alpha}\beta\Gamma(1+\alpha)}{\alpha\pi} s^{-1-\alpha} \leq \frac{C\beta}{\sqrt{\alpha}} s^{-1-\alpha}. \quad (11)$$

In a similar manner, (10) gives

$$-G'(s) \leq \frac{C\beta}{\sqrt{\alpha}} s^{-2-\alpha}, \quad G''(s) \leq \frac{C\beta}{\sqrt{\alpha}} s^{-3-\alpha}. \quad (12)$$

4. APPROXIMATION TO EIGENFUNCTIONS

Let n be a fixed positive integer and $\mu_n = \frac{n\pi}{2} - \frac{\beta\pi}{8}$. Our goal is to show that μ_n^α is close to λ_n . Note that $\mu_n \geq \frac{\pi}{4}$ and $\frac{n\pi}{4} \leq \mu_n \leq \frac{n\pi}{2}$.

We construct approximations $\tilde{\varphi}_n$ to eigenfunctions φ_n by combining shifted eigenfunctions for half-line, $F_{\mu_n}(1+x)$ and $F_{\mu_n}(1-x)$, and using the auxiliary function q given above in (5) to join them in a sufficiently smooth way. We let

$$\tilde{\varphi}_n(x) = q(-x)F_{\mu_n}(1+x) + (-1)^n q(x)F_{\mu_n}(1-x). \quad (13)$$

Lemma 1. *We have*

$$\|\mathcal{A}_D \tilde{\varphi}_n - \mu_n^\alpha \tilde{\varphi}_n\|_2 \leq \frac{C\beta}{\sqrt{\alpha}} \frac{1}{n}. \quad (14)$$

Proof. Note that we have

$$\begin{aligned} \tilde{\varphi}_n(x) - F_{\mu_n}(1+x) &= -(1 - q(-x))F_{\mu_n}(1+x) - (-1)^n q(x)F_{\mu_n}(1-x) \\ &= -q(x)(F_{\mu_n}(1+x) + (-1)^n F_{\mu_n}(1-x)) \\ &= q(x)(G_{\mu_n}(1+x) + (-1)^n G_{\mu_n}(1-x)) - \sin(\mu_n(1+x) + \frac{\beta\pi}{8})\mathbf{1}_{[1,\infty)}(x). \end{aligned}$$

Denote $h(x) = \sin(\mu_n(1+x) + \frac{\beta\pi}{8})\mathbf{1}_{[1,\infty)}(x)$ and $f(x) = G_{\mu_n}(1+x) + (-1)^n G_{\mu_n}(1-x)$, $g(x) = q(x)f(x)$. It follows that $\tilde{\varphi}_n(x) = F_{\mu_n}(1+x) + g(x) + h(x)$. For $x \in (-1, 0)$, we have $\mathcal{A}F_{\mu_n}(x) - \mu_n^\alpha F_{\mu_n}(x) = 0$ and $h(x) = 0$. Hence,

$$|\mathcal{A}\tilde{\varphi}_n(x) - \mu_n^\alpha \tilde{\varphi}_n(x)| \leq |\mathcal{A}g(x)| + |\mathcal{A}h(x)| + |\mu_n^\alpha g(x)|, \quad x \in (-1, 0). \quad (15)$$

We will now estimate each of the summands on the right hand side.

Using convexity of G , $-G'$ and G'' , and estimates (9), (11) and (12), we obtain that

$$\begin{aligned} \sup_{x \in (-\frac{1}{3}, \frac{1}{3})} |f(x)| &\leq G(\frac{2}{3}\mu_n) + G(\frac{4}{3}\mu_n) \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1-\alpha}, \\ \sup_{x \in (-\frac{1}{3}, \frac{1}{3})} |f'(x)| &\leq -\mu_n G'(\frac{2}{3}\mu_n) - \mu_n G'(\frac{4}{3}\mu_n) \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1-\alpha}, \\ \sup_{x \in (-\frac{1}{3}, \frac{1}{3})} |f''(x)| &\leq \mu_n^2 G''(\frac{2}{3}\mu_n) + \mu_n^2 G''(\frac{4}{3}\mu_n) \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1-\alpha}, \\ \int_0^\infty |f(x)| dx &\leq \int_0^\infty G_{\mu_n}(1+x) dx + \int_0^1 G_{\mu_n}(1-x) dx \\ &= \frac{1}{\mu_n} \int_0^\infty G(y) dy \leq \frac{C\beta}{\mu_n}. \end{aligned}$$

By (6) and (4),

$$|\mathcal{A}g(x)| \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1-\alpha} + C\alpha\beta^2 \mu_n^{-1}, \quad x \in (-1, 0). \quad (16)$$

Furthermore, $|g(x)| = 0$ for $x \in (-1, -\frac{1}{3})$, and

$$|\mu_n^\alpha g(x)| \leq \frac{1}{2} \mu_n^\alpha f(x) \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1}, \quad x \in (-\frac{1}{3}, 0). \quad (17)$$

Finally, for $x < 0$ we have the following estimate for the oscillatory integral

$$\begin{aligned} |\mathcal{A}h(x)| &= c_\alpha \left| \int_1^\infty \frac{\sin(\mu_n(1+y) + \frac{(2-\alpha)\pi}{8})}{|x-y|^{1+\alpha}} dy \right| \\ &\leq c_\alpha \int_1^{1+\pi/\mu_n} \frac{|\sin(\mu_n(1+y) + \frac{(2-\alpha)\pi}{8})|}{|x-1|^{1+\alpha}} dy \leq \frac{c_\alpha}{(1-x)^{1+\alpha} \mu_n} \leq 2\alpha\beta \mu_n^{-1}. \end{aligned} \quad (18)$$

Estimates (16)–(18) applied to (15) yield that

$$|\mathcal{A}\tilde{\varphi}_n(z) - \mu_n^\alpha \tilde{\varphi}_n(z)| \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1}, \quad z \in (-1, 0). \quad (19)$$

By symmetry, (19) also holds for $z \in (0, 1)$. Formula (14), with $\mathcal{A}_D \tilde{\varphi}_n$ understood in the pointwise sense, follows. It remains to prove that $\tilde{\varphi}_n$ is in the domain of \mathcal{A}_D . To this end, we will use the notion of the Green operator $G_D = \mathcal{A}_D^{-1}$. The reader is referred e.g. to [6] for formal definition and properties of G_D .

Since $\mathcal{A}\tilde{\varphi}_n$ is bounded on D , the function $\tilde{\varphi}_n - G_D \mathcal{A}\tilde{\varphi}_n$ is a bounded, continuous in D , weakly α -harmonic function in $D = (-1, 1)$ with zero exterior condition. Such a function is necessarily zero (see [4, 11]). It follows that $\tilde{\varphi}_n = G_D \mathcal{A}\tilde{\varphi}_n$, and hence $\tilde{\varphi}_n$ is in the $L^\infty(D)$ domain of \mathcal{A}_D . Since convergence in $L^\infty(D)$ is stronger than the one in $L^2(D)$, the proof is complete. \square

Lemma 2. *We have*

$$1 - \frac{C\beta}{n} \leq \|\tilde{\varphi}_n\|_2 \leq 1 + \frac{C\beta}{n}. \quad (20)$$

In particular, there is an absolute constant K such that $\|\tilde{\varphi}_n\|_2 \geq \frac{1}{2}$ for $n \geq K$.

Proof. First, note that by direct integration,

$$\left| \int_{-1}^1 \left(\left(\sin(\mu_n(x+1) + \frac{\beta\pi}{8}) \right)^2 - \frac{1}{2} \right) dx \right| \leq \frac{C\beta}{\mu_n}.$$

Using (13) and (9), we obtain the lower bound,

$$\begin{aligned} \|\tilde{\varphi}_n\|_2^2 &\geq \int_{-1}^1 \left(\sin(\mu_n(x+1) + \frac{\beta\pi}{8}) \right)^2 dx \\ &\quad - 4 \int_{-1}^1 |q(-x)G_{\mu_n}(x+1) \sin(\mu_n(x+1) + \frac{\pi}{8})| dx \geq 1 - \frac{C\beta}{\mu_n}. \end{aligned}$$

In a similar manner,

$$\|\tilde{\varphi}_n\|_2^2 \leq 1 + \frac{C}{\mu_n} + 4 \int_{-1}^1 (G(\mu_n(x+1)))^2 dx \leq 1 + \frac{C\beta}{\mu_n},$$

and the lemma is proved. \square

5. PROOF OF THEOREM 1

Since $\tilde{\varphi}_n \in L^2(D)$, we have $\tilde{\varphi}_n = \sum_j a_j \varphi_j$ for some a_j . Moreover, $\|\tilde{\varphi}_n\|_2^2 = \sum_j a_j^2$ and $\mathcal{A}_D \tilde{\varphi}_n = \sum_j \lambda_j a_j \varphi_j$. Let $\lambda_{k(n)}$ be the eigenvalue nearest to μ_n^α . Then

$$\|\mathcal{A}_D \tilde{\varphi}_n - \mu_n^\alpha \tilde{\varphi}_n\|_2^2 = \sum_{j=1}^{\infty} (\lambda_j - \mu_n^\alpha)^2 a_j^2 \geq (\lambda_{k(n)} - \mu_n^\alpha)^2 \sum_{j=1}^{\infty} a_j^2 = (\lambda_{k(n)} - \mu_n^\alpha)^2 \|\tilde{\varphi}_n\|_2^2.$$

By (14) and Lemma 2, it follows that for $n \geq K$,

$$|\lambda_{k(n)} - \mu_n^\alpha| \leq \frac{C\beta}{\sqrt{\alpha}} \frac{1}{n}. \quad (21)$$

This will enable us to derive a two-term asymptotic formula for λ_j .

Denote $\varepsilon = \frac{1}{2} \frac{\beta\pi}{8}$. We have

$$|(\mu_n \pm \varepsilon)^\alpha - \mu_n^\alpha| \geq \alpha \varepsilon \min((\mu_n - \varepsilon)^{\alpha-1}, (\mu_n + \varepsilon)^{\alpha-1}) \geq C\alpha \varepsilon n^{\alpha-1}. \quad (22)$$

Thus if $|\lambda_{k(n)} - \mu_n^\alpha| \leq C\alpha \varepsilon n^{\alpha-1}$, then $\lambda_n \in ((\mu_n - \varepsilon)^\alpha, (\mu_n + \varepsilon)^\alpha)$. By (21), this holds true if $n \geq K$ and

$$n \geq \left(\frac{C\beta}{\alpha^{3/2}\varepsilon} \right)^{\frac{1}{\alpha}} = (C'\alpha^{-3/2})^{1/\alpha}.$$

Therefore, $L_\alpha = \lceil (C\alpha^{-3/2})^{1/\alpha} \rceil$ (the constant here is chosen so that also $L_\alpha \geq K$) is such that for $n \geq L_\alpha$, each interval $((\mu_n - \varepsilon)^\alpha, (\mu_n + \varepsilon)^\alpha)$ contains an eigenvalue $\lambda_{k(n)}$. In particular $\lambda_{k(n)}$ are distinct for $n \geq L_\alpha$. We claim that there are less than L_α eigenvalues not included in the above class. As in [13], the key step will be the trace estimate.

Let J be the set of those $j > 0$ for which $j \neq k(n)$ for all $n \geq L_\alpha$. Denote by $p_t(x-y)$ and $p_t^D(x,y)$ the heat kernels for \mathcal{A} and \mathcal{A}_D respectively; we have $\hat{p}_t(\xi) = \exp(-t|\xi|^\alpha)$. For $t > 0$, we have (see e.g. [2, 12])

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_D \sum_{j=1}^{\infty} e^{-\lambda_j t} (\varphi_j(x))^2 dx = \int_D p_t^D(x,x) dx \leq \int_D p_t(0) dx = \frac{2}{\pi} \int_0^\infty e^{-ts^\alpha} ds.$$

In the last step, Fourier inversion formula was used. Hence,

$$\sum_{j \in J} e^{-\lambda_j t} = \sum_{j=1}^{\infty} e^{-\lambda_j t} - \sum_{n=L_\alpha}^{\infty} e^{-\lambda_{k(n)} t} \leq \frac{2}{\pi} \left(\int_0^{\infty} e^{-ts^\alpha} ds - \frac{\pi}{2} \sum_{n=L_\alpha}^{\infty} e^{-t(\mu_n + \varepsilon)^\alpha} \right).$$

The latter series is bounded below by the integral of e^{-ts^α} over $(\mu_{L_\alpha} + \varepsilon, \infty)$. Hence,

$$\sum_{j \in J} e^{-\lambda_j t} \leq \frac{2}{\pi} \int_0^{\mu_{L_\alpha} + \varepsilon} e^{-ts^\alpha} ds \leq \frac{2}{\pi} (\mu_{L_\alpha} + \varepsilon).$$

Taking the limit as $t \searrow 0$, we obtain that

$$\#J \leq \frac{2}{\pi} (\mu_{L_\alpha} + \varepsilon) = L_\alpha - \frac{\beta}{4} + \frac{2\varepsilon}{\pi}.$$

Since $\varepsilon < \frac{\beta\pi}{8}$, the right hand side is less than L_α , and the claim is proved.

By [8, 9], we have $\lambda_n \leq (n\pi/2)^\alpha$. It follows that for all $n < L_\alpha$, we have $\lambda_n < (\mu_{L_\alpha} - \varepsilon)^\alpha$, and so $J = \{1, 2, \dots, L_\alpha - 1\}$. We conclude that $k(n) = n$ for all $n \geq L_\alpha$. Theorem 1 now follows from (21).

6. FURTHER PROPERTIES OF EIGENVALUES AND EIGENFUNCTIONS

In this section three additional properties of φ_n and λ_n are studied. This part is modelled after [13], Section 10. A number of open problems is suggested at the end of the section.

Proposition 1 (cf. Lemma 3 and Corollary 4 in [13]). *There is a constant C such that,*

$$\begin{aligned} \|\tilde{\varphi}_n - \varphi_n\|_2 &\leq \frac{C(2-\alpha)}{n} && \text{when } \alpha \geq 1, \\ \|\tilde{\varphi}_n - \varphi_n\|_2 &\leq \frac{C(2-\alpha)}{\alpha^{3/2}n^\alpha} && \text{when } \alpha < 1. \end{aligned}$$

In particular, if $\varphi_n^(x) = \pm \cos(\mu_n x)$ for odd n and $\varphi_n^*(x) = \pm \sin(\mu_n x)$ for even n , then*

$$\begin{aligned} \|\varphi_n^* - \varphi_n\|_2 &\leq \frac{C(2-\alpha)}{\sqrt{n}} && \text{when } \alpha \geq \frac{1}{2}, \\ \|\varphi_n^* - \varphi_n\|_2 &\leq \frac{C(2-\alpha)}{\alpha^{3/2}n^\alpha} && \text{when } \alpha < \frac{1}{2}. \end{aligned}$$

for some constant C .

Proof. Fix $n \geq L_\alpha + 1$ and $\varepsilon = \frac{1}{2} \frac{\beta\pi}{8}$, and write, as in the previous section, $\tilde{\varphi}_n = \sum_j a_j \varphi_j$. Changing the sign of φ_n if necessary, we may assume that $a_n > 0$. As in (22), for $j \neq n$ we have $|\lambda_j - \mu_n^\alpha| \geq C\alpha n^{\alpha-1}$. Hence,

$$\|\mathcal{A}_D \tilde{\varphi}_n - \mu_n^\alpha \tilde{\varphi}_n\|_2^2 = \sum_{j=1}^{\infty} (\lambda_j - \mu_n^\alpha)^2 a_j^2 \geq C(\alpha n^{\alpha-1})^2 \sum_{j \neq n} a_j^2.$$

By (14), we obtain that

$$\|\tilde{\varphi}_n - a_n \varphi_n\|_2^2 = \sum_{j \neq n} a_j^2 \leq C \left(\frac{\beta}{\sqrt{\alpha}} \frac{1}{n} \right)^2 \frac{1}{(\alpha n^{\alpha-1})^2} = C \left(\frac{\beta}{\alpha^{3/2} n^\alpha} \right)^2. \quad (23)$$

Note that

$$\|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2 \varphi_n\|_2 \leq \|\tilde{\varphi}_n - a_n \varphi_n\|_2 + |a_n - \|\tilde{\varphi}_n\|_2|.$$

But $|a_n - \|\tilde{\varphi}_n\|_2|^2 \leq \|\tilde{\varphi}_n\|_2^2 - a_n^2 = \|\tilde{\varphi}_n - a_n \varphi_n\|_2^2$. Hence, using also (20), we obtain that

$$\|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2 \varphi_n\|_2 \leq \frac{2C\beta}{\alpha^{3/2}n^\alpha}. \quad (24)$$

Finally, by (23),

$$\|\tilde{\varphi}_n - \varphi_n\|_2 \leq \|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2 \varphi_n\|_2 + |\|\tilde{\varphi}_n\|_2 - 1| \leq \frac{2C\beta}{\alpha^{3/2}n^\alpha} + \frac{C\beta}{n}.$$

We have thus proved the first part of the proposition. The second statement is a simple consequence of the first one, the identity $\tilde{\varphi}_n(x) = \varphi_n^*(x) - G(\mu_n(1+x)) \pm G(\mu_n(1-x))$, and the estimate $\|G\|_2^2 \leq \|G\|_1 \|G\|_\infty \leq C\beta^2$. \square

Proposition 2 (cf. Corollary 5 in [13]). *If $\alpha \geq \frac{1}{2}$, then the eigenfunctions $\varphi_n(x)$ are bounded uniformly in $n \geq 1$ and $x \in D$.*

Proof. Let $P_t^D = \exp(-t\mathcal{A}_D)$ ($t > 0$) be the heat semigroup for \mathcal{A}_D (or transition semigroup of the symmetric α -stable process in D), and let $p_t^D(x, y)$ be the corresponding heat kernel (or transition density). It is well known that $p_t^D(x, y) \leq p_t(y - x)$, where $p_t(x)$ is the heat kernel for \mathcal{A} and $\hat{p}_t(\xi) = \exp(-t|\xi|^\alpha)$; see e.g. [5].

By Cauchy-Schwarz inequality and Plancherel's theorem, we obtain

$$\begin{aligned} e^{-\lambda_n t} |\varphi_n(x)| &\leq |P_t^D(\varphi_n - \tilde{\varphi}_n)(x)| + |P_t^D \tilde{\varphi}_n(x)| \\ &\leq \sqrt{\int_{-\infty}^{\infty} (p_t(x - y))^2 dy} \|\varphi_n - \tilde{\varphi}_n\|_2 + \|\tilde{\varphi}_n\|_\infty \\ &= \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2t|z|^\alpha} dz} \|\varphi_n - \tilde{\varphi}_n\|_2 + \|F\|_\infty \\ &\leq 2\sqrt{\Gamma(1 + 1/\alpha)} (2t)^{-1/(2\alpha)} \|\varphi_n - \tilde{\varphi}_n\|_2 + 2, \end{aligned}$$

Let $t = 1/\lambda_n$. Then $e^{-\lambda_n t} = 1/e$ and $t^{-1/(2\alpha)} \leq \lambda_n^{1/(2\alpha)} \leq C\sqrt{n}$. If $n \geq L_\alpha$ and $\alpha \geq \frac{1}{2}$, then also $\|\varphi_n - \tilde{\varphi}_n\|_2 \leq C/\sqrt{n}$, and finally $|\varphi_n(x)| \leq C$. Since each φ_n is in $L^\infty(D)$, the proof is complete. \square

Proposition 3 (cf. Theorem 6 in [13]). *If $\alpha \geq 1$, then the eigenvalues λ_n are simple.*

Proof. Let us write $\lambda_{n,\alpha}$ for λ_n in this proof. Since $(\lambda_{n,\alpha})^{1/\alpha}$ is increasing in α , we have

$$(\lambda_{n,\alpha})^{1/\alpha} \leq (\lambda_{n,2})^{1/2} = \frac{n\pi}{2}.$$

By Theorem 6 in [13], for $n \geq 3$ we have

$$\frac{(n+1)\pi}{2} - \frac{\pi}{8} - \frac{\pi}{10} < \lambda_{n+1,1} \leq (\lambda_{n+1,\alpha})^{1/\alpha}.$$

Therefore, $\lambda_{n,\alpha} < \lambda_{n+1,\alpha}$, except perhaps $n = 1$ or $n = 2$. But a similar argument works also for $n = 1$ and $n = 2$, since by [1] we have

$$\frac{\pi}{2} < 2 \leq \lambda_{2,1}, \quad \text{and} \quad \pi < 3.83 < \lambda_{3,1}.$$

The proof is complete. \square

Numerical experiments suggest that φ_n are uniformly bounded also for $\alpha < \frac{1}{2}$. Furthermore, it would be interesting to obtain an upper estimate of $\sup_n \|\varphi_n\|_\infty$, and in particular, to find its behavior when α approaches 0. Finally, as stated in the introduction, better bounds for λ_n may yield simplicity of eigenvalues also when $\alpha < 1$.

7. NUMERICAL BOUNDS FOR EIGENVALUES

No general *efficient* algorithm giving mathematically correct numerical bounds for λ_n is known to the author. For $\alpha = 1$, a satisfactory method (an application of Rayleigh-Ritz and Weinstein-Aronszajn methods) is described in [13]. For general α , even approximation of λ_n is difficult: all known methods converge rather slowly, and thus the computation of eigenvalues of very large matrices is required. In this section a method for obtaining a lower bound for λ_n is described. It shares the main drawbacks of many related algorithms: compared to the technique applied in [15], it converges slowly, and it suffers large errors as α approaches 2. On the other hand, the method presented below gives mathematically correct lower bounds, and there is no error estimate for the numerical scheme of [15]. At the end of the section, a somewhat similar method for the upper bound for λ_1 is given. It gives satisfactory results for large α , but deteriorates as α gets close to 0.

It should be pointed out that in many particular cases (α close to 2 or n large), the bound $\frac{1}{2}(\frac{n\pi}{2})^\alpha \leq \lambda_n \leq (\frac{n\pi}{2})^\alpha$ of [7, 9] is sharper than the estimates obtained below, unless extremely large matrices are used. Also, good numerical estimates of λ_n are available for $\alpha = 1$ due to [13]. By the monotonicity of $(\lambda_n)^{1/\alpha}$ in α , this gives a lower bound for λ_n when $\alpha \in (1, 2)$ and an upper bound for $\alpha \in (0, 1)$. Finally, a good estimate of λ_1 can be found in [1]. For a comparison of the above, see Table 2.

Our method for the lower bound works for fractional Laplace operator in an arbitrary bounded open set $D \subseteq \mathbf{R}^d$ (in fact, it can be easily extended to more general pseudo-differential operators, or Lévy processes). Fix $\varepsilon > 0$ and let $\{I_k : k \in \mathbf{Z}^d\}$ be the partition of \mathbf{R}^d into cubes $I_k = \prod_{j=1}^d [k_j\varepsilon, (k_j + 1)\varepsilon]$, $k \in \mathbf{Z}^d$. Let $K_\varepsilon \subseteq \mathbf{Z}^d$ be the set of those $k \in \mathbf{Z}^d$ for which I_k intersects D , and let D_ε be the interior of $\bigcup_{k \in K_\varepsilon} I_k$. Note that $D \subseteq D_\varepsilon$.

The definition of $\mathcal{A} = (-\Delta)^{\alpha/2}$ in higher dimension is similar to (3): for smooth bounded functions we have

$$\mathcal{A}f(x) = c_{d,\alpha} \text{pv} \int_{\mathbf{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} dy, \quad x \in \mathbf{R}^d,$$

where $c_{d,\alpha} = 2^\alpha \Gamma((d + \alpha)/2) / (\pi^{d/2} |\Gamma(-\frac{\alpha}{2})|)$. Fractional Laplace operator in D with zero exterior condition, denoted \mathcal{A}_D , is defined as in dimension one. Below we denote by λ_n the eigenvalues of \mathcal{A}_D . By domain monotonicity of λ_n , the eigenvalues for D are not less than than the eigenvalues of its superset D_ε . For notational convenience, we assume that $D = D_\varepsilon$.

The Dirichlet form $\mathcal{E}(f, f)$ corresponding to \mathcal{A}_D is given by

$$\mathcal{E}(f, f) = \frac{c_{d,\alpha}}{2} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} dx dy, \quad f \in L^2(D).$$

α	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
0.01	0.9966	1.0086	1.0137	1.0171	1.0196	1.0217	1.0234	1.0248	1.0261	1.0273
	0.9943 ¹	0.5057 ²	0.5078 ²	0.5092 ²	0.5104 ²	0.5113 ²	0.5121 ²	0.5128 ²	0.5134 ²	0.5139 ²
	0.9976	1.0086	1.0138	1.0172	1.0198	1.0218	1.0235	1.0250	1.0263	1.0274
	0.9966 ⁴	1.0087 ⁴	1.0137 ⁴	1.0172 ⁴	1.0197 ⁴	1.0218 ⁴	1.0235 ⁴	1.0250 ⁴	1.0263 ⁴	1.0274 ⁴
	13.5210	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	0.9974 ¹	1.0102 ³	1.0148 ³	1.0179 ³	1.0203 ³	1.0223 ³	1.0239 ³	1.0254 ³	1.0266 ³	1.0277 ³
0.1	0.9724	1.0919	1.1469	1.1863	1.2159	1.2405	1.2611	1.2791	1.2950	1.3094
	0.9513 ¹	0.5606 ²	0.5838 ²	0.6008 ²	0.6144 ²	0.6257 ²	0.6354 ²	0.6440 ²	0.6516 ²	0.6585 ²
	0.9809	1.0913	1.1477	1.1867	1.2167	1.2412	1.2620	1.2802	1.2962	1.3107
	0.9726 ⁴	1.0922 ⁴	1.1473 ⁴	1.1868 ⁴	1.2165 ⁴	1.2413 ⁴	1.2620 ⁴	1.2802 ⁴	1.2962 ⁴	1.3107 ⁴
	1.8351	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	0.9786 ¹	1.1067 ³	1.1575 ³	1.1941 ³	1.2226 ³	1.2462 ³	1.2664 ³	1.2840 ³	1.2997 ³	1.3138 ³
0.2	0.9572	1.1960	1.3182	1.4093	1.4801	1.5402	1.5915	1.6373	1.6780	1.7154
	0.9181 ¹	0.6286 ²	0.6817 ²	0.7221 ²	0.7550 ²	0.7831 ²	0.8076 ²	0.8294 ²	0.8492 ²	0.8673 ²
	0.9712	1.1948	1.3199	1.4102	1.4819	1.5420	1.5939	1.6399	1.6812	1.7188
	0.9575 ⁴	1.1965 ⁴	1.3191 ⁴	1.4105 ⁴	1.4817 ⁴	1.5421 ⁴	1.5938 ⁴	1.6400 ⁴	1.6811 ⁴	1.7188 ⁴
	1.2376	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	0.9675 ¹	1.2247 ³	1.3398 ³	1.4258 ³	1.4947 ³	1.5530 ³	1.6036 ³	1.6485 ³	1.6890 ³	1.7260 ³
0.5	0.9692	1.5991	2.0247	2.3809	2.6862	2.9618	3.2118	3.4443	3.6608	3.8654
	0.8862 ²	0.8862 ²	1.0854 ²	1.2533 ²	1.4012 ²	1.5349 ²	1.6579 ²	1.7724 ²	1.8799 ²	1.9816 ²
	0.9908	1.5977	2.0306	2.3862	2.6954	2.9725	3.2259	3.4608	3.6808	3.8883
	0.9701 ⁴	1.6015 ⁴	2.0288 ⁴	2.3871 ⁴	2.6947 ⁴	2.9728 ⁴	3.2255 ⁴	3.4610 ⁴	3.6805 ⁴	3.8883 ⁴
	1.0002	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	0.9863 ¹	1.6598 ³	2.0777 ³	2.4274 ³	2.7314 ³	3.0055 ³	3.2562 ³	3.4892 ³	3.7074 ³	3.9136 ³
1	1.1516	2.7343	4.2756	5.8236	7.3584	8.8919	10.4166	11.9382	13.4528	14.9636
	1.1577 ³	2.7547 ³	4.3168 ³	5.8921 ³	7.4601 ³	9.0328 ³	10.6022 ³	12.1741 ³	13.7441 ³	15.3155 ³
	1.1781	2.7489	4.3197	5.8905	7.4613	9.0321	10.6029	12.1737	13.7445	15.3153
	1.1577 ⁴	2.7545 ⁴	4.3164 ⁴	5.8916 ⁴	7.4594 ⁴	9.0319 ⁴	10.6012 ⁴	12.1729 ⁴	13.7427 ⁴	15.3140 ⁴
	1.1608	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	1.1578 ³	2.7548 ³	4.3169 ³	5.8922 ³	7.4602 ³	9.0329 ³	10.6023 ³	12.1742 ³	13.7442 ³	15.3156 ³
1.5	1.5139	4.7367	8.8817	13.7668	19.2502	25.2613	31.7334	38.6263	45.8996	53.5266
	1.3293 ¹	4.5721 ³	8.9689 ³	14.3024 ³	20.3762 ³	27.1479 ³	34.5222 ³	42.4772 ³	50.9536 ³	59.9375 ³
	1.6114	5.0545	9.5970	15.0171	21.1905	28.0344	35.4886	43.5067	52.0514	61.0922
	1.5971 ⁴	5.0586 ⁴	9.5921 ⁴	15.0154 ⁴	21.1846 ⁴	28.0289 ⁴	35.4800 ⁴	43.4972 ⁴	52.0392 ⁴	61.0786 ⁴
	1.5989	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	1.6224 ¹	5.5684 ²	10.2297 ²	15.7497 ²	22.0108 ²	28.9339 ²	36.4609 ²	44.5467 ²	53.1550 ²	62.2558 ²
1.8	1.4483	5.1149	10.4447	17.2231	25.2907	34.5448	44.8969	56.2813	68.6385	81.9210
	1.6765 ¹	6.1965 ³	13.9088 ³	24.3496 ³	37.2347 ³	52.5393 ³	70.1002 ³	89.9057 ³	111.8432 ³	135.9060 ³
	2.0555	7.5003	15.8014	26.7233	40.1148	55.8658	73.8905	94.1188	116.4923	140.9605
	2.0481 ⁴	7.5007 ⁴	15.7948 ⁴	26.7156 ⁴	40.1012 ⁴	55.8481 ⁴	73.8661 ⁴	94.0884 ⁴	116.4541 ⁴	140.9145 ⁴
	2.0501	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	2.0777 ¹	7.8501 ²	16.2868 ²	27.3353 ²	40.8472 ²	56.7138 ²	74.8501 ²	95.1871 ²	117.6664 ²	142.2381 ²
1.9	1.0353	3.7704	7.8734	13.1989	19.6379	27.1159	35.5691	44.9481	55.2082	66.3127
	1.8273 ¹	6.8573 ³	16.0993 ³	29.0750 ³	45.5221 ³	65.4737 ³	88.7686 ³	115.4333 ³	145.3521 ³	178.5468 ³
	2.2477	8.5942	18.7177	32.4615	49.7204	70.4157	94.4848	121.8754	152.5433	186.4500
	2.2432 ⁴	8.5926 ⁴	18.7101 ⁴	32.4503 ⁴	49.7021 ⁴	70.3905 ⁴	94.4503 ⁴	121.8313 ⁴	152.4878 ⁴	186.3822 ⁴
	2.2455	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	2.2748 ¹	8.8021 ²	19.0178 ²	32.8505 ²	50.1962 ²	70.9766 ²	95.1293 ²	122.6024 ²	153.3517 ²	187.3389 ²
1.99	0.1474	0.5494	1.1671	1.9816	2.9788	4.1482	5.4811	6.9705	8.6101	10.3944
	1.9816 ¹	7.5121 ³	18.3642 ³	34.1070 ³	54.5469 ³	79.8163 ³	109.7856 ³	144.5508 ³	184.0144 ³	228.2517 ³
	2.4441	9.7330	21.8288	38.7113	60.3666	86.7839	117.9546	153.8713	194.5275	239.9178
	2.4427 ⁴	9.7293 ⁴	21.8200 ⁴	38.6960 ⁴	60.3426 ⁴	86.7495 ⁴	117.9077 ⁴	153.8100 ⁴	194.4500 ⁴	239.8220 ⁴
	2.4452	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	2.4563 ²	9.7573 ²	21.8651 ²	38.7595 ²	60.4267 ²	86.8560 ²	118.0385 ²	153.9670 ²	194.6351 ²	240.0373 ²

¹ See [1]. ² See [7]. ³ Combination of [13] with monotonicity in α . ⁴ See [15].

TABLE 2. Comparison of bounds and approximations to λ_n . Each cell contains six numbers: lower bound $\lambda_{n,\varepsilon}$ with $\varepsilon = \frac{1}{2500}$, the best lower bound known before, approximation $(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8})^\alpha$, numerical approximation of [15], upper bound $\lambda_{1,\varepsilon}^*$, the best upper bound known before. The better estimates are printed in color.

As usual, $f \in L^2(D)$ is extended to \mathbf{R}^d so that $f(x) = 0$ for $x \in \mathbf{R}^d \setminus D$. For $k \in \mathbf{Z}^d$, denote

$$\|k\| = \sqrt{\sum_{j=1}^d (|k_j| + 1)^2}.$$

When $x \in I_k$, $y \in I_l$, $k, l \in \mathbf{Z}^d$, we have $|x - y| \leq \varepsilon \|k - l\|$. We define

$$\nu_k = \|k\|^{-d-\alpha}, \quad \bar{\nu} = \sum_{k \in \mathbf{Z}^d} \nu_k,$$

and

$$\mathcal{E}_\varepsilon(f, f) = \frac{c_{d,\alpha} \varepsilon^{-d-\alpha}}{2} \sum_{k,l \in \mathbf{Z}^d} \nu_{k-l} \int_{I_k} \int_{I_l} (f(x) - f(y))^2 dx dy.$$

Clearly, $\mathcal{E}_\varepsilon(f, f) \leq \mathcal{E}(f, f)$. By Rayleigh-Ritz variational principle, the eigenvalues λ_n are bounded below by the sequence $\lambda_{n,\varepsilon}$ of eigenvalues of the operator corresponding to the Dirichlet form \mathcal{E}_ε . More precisely, $\lambda_{n,\varepsilon}$ are defined in the usual way,

$$\lambda_{n,\varepsilon} = \inf \left\{ \sup \{ \mathcal{E}_\varepsilon(f, f) : f \in L, \|f\|_2 = 1 \} : L < L^2(D), \dim L = n \right\}.$$

Here ' $L < L^2(D)$ ' means that L is a linear subspace of $L^2(D)$.

For $f \in L^2(D)$ and $k \in \mathbf{Z}^d$, let $f_k = \varepsilon^{-d} \int_{I_k} f(x) dx$, and define f^* to be equal to f_k on I_k . Hence $f^* \in L^2(D)$ is the orthogonal projection of f onto the space of functions constant on each I_k , and $\int_{I_k} f^*(x) dx = \int_{I_k} f(x) dx$. In particular, $\|f\|_2^2 = \|f^*\|_2^2 + \|f - f^*\|_2^2$. Furthermore,

$$\begin{aligned} \mathcal{E}_\varepsilon(f, f) &= \frac{c_{d,\alpha} \varepsilon^{-d-\alpha}}{2} \sum_{k,l \in \mathbf{Z}^d} \nu_{k-l} \int_{I_k} \int_{I_l} ((f(x))^2 - 2f(x)f(y) + (f(y))^2) dx dy \\ &= c_{d,\alpha} \varepsilon^{-\alpha} \left(\bar{\nu} \|f\|_2^2 - \varepsilon^d \sum_{k,l \in \mathbf{Z}^d} \nu_{k-l} f_k f_l \right). \end{aligned}$$

Comparing this with a similar formula for f^* , we obtain that

$$\mathcal{E}_\varepsilon(f, f) = \mathcal{E}_\varepsilon(f^*, f^*) + c_{d,\alpha} \varepsilon^{-\alpha} \bar{\nu} \|f - f^*\|_2^2$$

This shows that the two orthogonal subspaces, $\{f \in L^2(D) : f^* = 0\}$ and $\{f \in L^2(D) : f^* = f\}$, are invariant under the action of the operator corresponding to \mathcal{E}_ε . The former subspace is in fact its eigenspace, corresponding to the eigenvalue $c_{d,\alpha} \varepsilon^{-\alpha}$. The latter one is finite-dimensional, and the action of \mathcal{E}_ε in the basis of normalized indicators of I_k , $k \in K_\varepsilon$, is given by the following matrix V : if κ be a bijection between $\{1, 2, \dots, |K_\varepsilon|\}$ and K_ε , then $V_{p,q} = c_{d,\alpha} \varepsilon^{-\alpha} (\delta_{p,q} \nu^* - \nu_{\kappa(p) - \kappa(q)})$.

We conclude that the sequence $\lambda_{n,\varepsilon}$ starts with those eigenvalues of the matrix V which are less than $c_{d,\alpha} \varepsilon^{-\alpha} \bar{\nu}$, which are followed by the constant $c_{d,\alpha} \varepsilon^{-\alpha} \bar{\nu}$. This gives the lower bound for the eigenvalues λ_n for an arbitrary open bounded set D . Note that replacing $\bar{\nu}$ by a smaller number gives smaller lower bounds $\lambda_{n,\varepsilon}$, hence the series defining $\bar{\nu}$ should be approximated from below.

When $D = (-1, 1) \subseteq \mathbf{R}$ and $\varepsilon = \frac{2}{N}$, then $\bar{\nu} = 2\zeta(1 + \alpha) - 1$, where ζ is the Riemann zeta function. Furthermore, in this case V is a Toeplitz matrix with the symbol

$$\begin{aligned} \frac{2c_{d,\alpha}}{\varepsilon^\alpha} \left(\zeta(1 + \alpha) - \sum_{k=0}^{\infty} \frac{\cos(kx)}{(1+k)^{1+\alpha}} \right) &= \frac{2c_{d,\alpha}}{\varepsilon^\alpha} \left(\zeta(1 + \alpha) - \operatorname{Re} \left(\frac{\operatorname{Li}_{1+\alpha}(e^{ix})}{e^{ix}} \right) \right) \\ &= \frac{2c_{d,\alpha}}{\varepsilon^\alpha} \left(\zeta(1 + \alpha) - \frac{1}{1 + \alpha} \int_0^\infty \frac{t^\alpha (e^t - \cos x)}{e^{2t} - 2e^t \cos x + 1} dt \right). \end{aligned}$$

The right hand side is easily checked to be increasing in $x \in [0, \pi]$, and so it attains its maximum for $x = \pi$. The symbol of V , and hence the eigenvalues of V , are therefore bounded above by $2c_{d,\alpha}\varepsilon^{-\alpha}(\zeta(1 + \alpha) - \operatorname{Li}_{1+\alpha}(-1)) = 2^{1-\alpha}c_{d,\alpha}\varepsilon^{-\alpha}\zeta(1 + \alpha) \leq c_{d,\alpha}\varepsilon^{-\alpha}\bar{\nu}$. It follows that all N eigenvalues of V are included in the sequence $\lambda_{n,\varepsilon}$.

In higher dimensions, $\bar{\nu}$ can only be computed by approximating numerically a d -dimensional infinite series. We summarize the results of this section in the following two results.

Proposition 4. *Let $D = (-1, 1)$, $N > 0$ and $\varepsilon = 2/N$. Let V be a $N \times N$ Toeplitz matrix with entries*

$$\begin{aligned} V_{p,q} &= -\frac{c_\alpha}{\varepsilon^\alpha} \frac{1}{(|p-q|+1)^{d+\alpha}}, & p, q = 1, 2, \dots, N, \ p \neq q; \\ V_{p,p} &= \frac{2c_\alpha(\zeta(1 + \alpha) - 1)}{\varepsilon^\alpha}, & p = 1, 2, \dots, N. \end{aligned}$$

Define $\lambda_{n,\varepsilon}$ to be the n -th smallest eigenvalue of V when $n \leq N$, and $\lambda_{n,\varepsilon} = c_\alpha\varepsilon^{-\alpha}(2\zeta(1 + \alpha) - 1)$ otherwise. Then the eigenvalues λ_n of \mathcal{A}_D satisfy $\lambda_n \geq \lambda_{n,\varepsilon}$.

Proposition 5. *Let $D \subseteq \mathbf{R}^d$ be an open set in \mathbf{R}^d , and let $\varepsilon > 0$. Let K_ε be the set of those $k \in \mathbf{Z}^d$ for which $D \cap \prod_{j=1}^d [k_j\varepsilon, (k_j + 1)\varepsilon]$ is nonempty, and let $\kappa : \{1, 2, \dots, |K_\varepsilon|\} \rightarrow K_\varepsilon$ be the enumeration of elements of K_ε . Finally, let*

$$\bar{\nu} = \sum_{k \in \mathbf{Z}^d} \|k\|^{-d-\alpha}, \quad \text{where} \quad \|k\| = \sqrt{\sum_{j=1}^d (|k_j| + 1)^2}.$$

Define a $|K_\varepsilon| \times |K_\varepsilon|$ matrix V with entries

$$\begin{aligned} V_{p,q} &= -\frac{c_{d,\alpha}}{\varepsilon^\alpha} \|\kappa(p) - \kappa(q)\|^{-d-\alpha}, & p, q = 1, 2, \dots, |K_\varepsilon|, \ p \neq q; \\ V_{p,p} &= \frac{c_{d,\alpha}}{\varepsilon^\alpha} (\bar{\nu} - d^{-(d+\alpha)/2}), & p = 1, 2, \dots, N. \end{aligned}$$

Let $\lambda_{n,\varepsilon}$ be the n -th smallest eigenvalue of V if $n \leq N$ and this eigenvalue does not exceed $c_{d,\alpha}\varepsilon^{-\alpha}\bar{\nu}$, and $\lambda_{n,\varepsilon} = c_{d,\alpha}\varepsilon^{-\alpha}\bar{\nu}$ otherwise. Then the eigenvalues λ_n of \mathcal{A}_D satisfy $\lambda_n \geq \lambda_{n,\varepsilon}$.

The lower bounds $\lambda_{n,\varepsilon}$ for the interval $D = (-1, 1)$ are presented in Table 2 above. In higher dimensions, the complexity of computations increases dramatically. For example, a unit disk $B(0, 1)$ or a square $[-1, 1]^2$ with $\varepsilon = \frac{1}{25}$ require handling matrices larger than 2000×2000 . Some results for these two cases are given in Tables 3 and 4.

In principle, the upper bound is much more difficult. The above approach can be modified to give an upper bound for λ_1 whenever the Green function for D can be computed. For

α	λ_1 (LB)		λ_1 (UB)	λ_2 (LB)		λ_2 (UB)
0.1	1.0308	0.5230 ¹	1.0462 ¹	1.0880	0.5415 ¹	1.0831 ¹
0.2	1.0506	0.5472 ¹	1.0946 ¹	1.1691	0.5865 ¹	1.1731 ¹
0.5	1.1587	0.6266 ¹	1.2534 ¹	1.4908	0.7452 ¹	1.4905 ¹
1	1.3844	0.7853 ¹	1.5708 ¹	2.1807	1.1107 ¹	2.2215 ¹
1.5	1.4135	0.9843 ¹	1.9688 ¹	2.6029	1.6554 ¹	3.3110 ¹
1.8	0.9167	1.1271 ¹	2.2544 ¹	1.8164	2.1033 ¹	4.2068 ¹
1.9	0.5427	1.1792 ¹	2.3585 ¹	1.0984	2.2781 ¹	4.5563 ¹

¹ See [7].

TABLE 3. Comparison of estimates of λ_n for a square $[-1, 1]^2$. LB and UB mean lower bounds and upper bounds respectively. Estimates of this section are given in roman font, best numerical estimates known before are typeset in slanted font. Better estimates are printed in color.

α	λ_1 (LB)		λ_1 (UB)	λ_2 (LB)		λ_2 (UB)
0.1	1.0381	1.0157 ¹	6.6198	1.0953	0.5718 ²	1.1609 ²
0.2	1.0655	1.0396 ¹	3.8878	1.1849	0.6541 ²	1.3476 ²
0.5	1.1986	1.1618 ¹	2.5081	1.5404	0.9787 ²	2.1079 ²
1	1.4734	1.5707 ¹	2.7588	2.3201	1.9158 ²	4.4429 ²
1.5	1.5387	2.3891 ¹	4.0668	2.8379	3.7502 ²	9.3648 ²
1.8	1.0087	3.2210 ¹	5.5014	2.0045	5.6114 ²	14.6487 ²
1.9	0.5990	3.5834 ¹	6.1369	1.2165	6.4182 ²	17.0045 ²

¹ See [1]. ² See [7].

TABLE 4. Comparison of estimates of λ_n for a unit disk. LB and UB mean lower bounds and upper bounds respectively. Estimates of this section are given in roman font, best numerical estimates known before are typeset in slanted font. Better estimates are printed in color.

the fractional Laplace operator, this is the case when D is a ball. By a scaling property, it is enough to consider $D = B(0, 1)$.

Let G_D be the Green function of D , $G_D(x, y) = \int_0^\infty p_t^D(x, y) dt$, where p_t^D is the heat kernel for \mathcal{A}_D (see the proof of Proposition 2). The Green function is the kernel of \mathcal{A}_D^{-1} . M. Riesz proved that

$$\begin{aligned}
G_D(x, y) &= \frac{\Gamma(\frac{d}{2})|x - y|^{\alpha-d}}{2^\alpha \pi^{d/2} \Gamma(\frac{\alpha}{2})^2} \int_0^{\frac{(1-x^2)(1-y^2)}{|x-y|^2}} \frac{s^{\alpha/2-1}}{(1+s)^{d/2}} ds \\
&= \frac{\Gamma(\frac{d}{2})(1-x^2)^{\alpha/2}(1-y^2)^{\alpha/2}}{2^\alpha \pi^{d/2} \Gamma(\frac{\alpha}{2}) \Gamma(1+\frac{\alpha}{2}) |x-y|^d} {}_2F_1\left(\frac{\alpha}{2}, \frac{d}{2}; 1+\frac{\alpha}{2}; -\frac{(1-x^2)(1-y^2)}{|x-y|^2}\right).
\end{aligned}$$

Since the eigenvalues of \mathcal{A}_D^{-1} are λ_n^{-1} , we have

$$\frac{1}{\lambda_1} = \sup \left\{ \int_D \int_D G_D(x, y) f(x) f(y) dx dy : f \in L^2(D), \|f\|_2 = 1 \right\}.$$

Since $G_D(x, y)$ is nonnegative, we may restrict the supremum to nonnegative functions only. Hence, whenever $g(x, y) \leq G_D(x, y)$, we have

$$\lambda_1 \leq \left(\sup \left\{ \int_D \int_D g(x, y) f(x) f(y) dx dy : f \in L^2(D), \|f\|_2 = 1 \right\} \right)^{-1}.$$

For $k, l \in \mathbf{Z}^d$, let $g_{k,l}$ be the infimum of $G_D(u, v)$ over $u \in I_k$ and $v \in I_l$. When $x \in I_k$, $y \in I_l$, we choose $g(x, y) = g_{k,l}$. Hence, λ_1 is bounded above by $\lambda_{1,\varepsilon}^*$, the reciprocal of the largest eigenvalue of the matrix U with entries $U_{i,j} = \varepsilon^d g_{\kappa(i),\kappa(j)}$.

The results for $D = (-1, 1) \subseteq \mathbf{R}$ and some values of α are given in Table 2. Estimates for the unit disk and the square $[-1, 1]^2$ are given in Tables 3 and 4. Noteworthy, for the unit disk and $\varepsilon = \frac{1}{25}$, the estimate $\lambda_{1,\varepsilon}^*$ is worse than the one obtained in [1] using analytical methods.

Acknowledgments. I would like to thank Krzysztof Bogdan and Tadeusz Kulczycki for helpful discussion and valuable suggestions.

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